Yang-Mills = theories of light, self-interacting, spin-1 particles

study the massless limit first

From L2 we recall that spin-1 massless particles have non-vanishing overlap with fields A_μ only if it is a Lorentz vector up to gauge transf.

$$(1)\quad \begin{cases} U(\Lambda)\, A_\mu(x)\, U^\dagger(\Lambda) = (\Lambda^{-1})_\mu{}^\nu A_\nu(\Lambda x) + \partial_\mu \Omega \\ \langle 0|\, A_\mu(x)\, |\bar{p}\,\lambda, s=1\ M=0\rangle = \varepsilon^\lambda_\mu(\bar{p}) \end{cases}$$

$$\Downarrow$$

$$(2)\quad \begin{cases} \epsilon^\lambda_\mu(p) = \Lambda_\mu{}^\nu\, \varepsilon^\lambda_\nu(\bar{p}) + \# p_\mu \\ \Lambda_\mu{}^\nu\, \epsilon^\lambda_\nu(p) = e^{i\lambda\theta_W}\left(\varepsilon^\lambda_\mu(\Lambda p) + \# \Lambda p\right) \end{cases}$$

The way Lorentz invariance is recovered is by demanding gauge invariance that is that the action is invariant under

$$(3)\qquad \boxed{A_\mu \to A_\mu + \partial_\mu \Omega}$$

Gauge transform. 1-massless spin-1

and physical observables such as S-matrix invariant under

(4) $$\boxed{\epsilon^{\lambda}_{\mu}(p) \to \epsilon^{\lambda}_{\mu}(p) + p_\mu}$$ Gauge redundancy polarizations

More precisely, two polarizations that differ by a gauge transformation should consider equivalent choices (e.g. $\epsilon_\mu = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ \pm i \\ 0 \end{pmatrix}$ and $\frac{1}{\sqrt{2}}\begin{pmatrix} E \\ 1 \\ \pm i \\ E \end{pmatrix}$ are equivalent polarizations)

— Classical Yang-Mills —

We have also seen that $F_{\mu\nu} = \partial_{[\mu} A_{\nu]} = (1,0)\oplus(0,1)$ is automatically gauge invariant so that

(5) $$\boxed{S_{\text{free}}[A] = -\frac{1}{4}\int_{\text{Mink.}} d^4x \; F_{\mu\nu} F^{\mu\nu}}$$ action free photon

(In this lecture I'm always in Minkowski sign.)

is the correct choice for the free kinetic term.

The generalization to several "species" or "flavors" of different spin-1 massless particles of the free term is just

(6) $$S_{\text{free}}[A^a] = -\frac{1}{4}\int d^4x \; F^a_{\mu\nu} F^a_{\mu\nu}$$ ← sum over "a" understood

$$\left(F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu = \partial_{[\mu} A^a_{\nu]}\right)$$

For later convenience it's actually useful, although by no means essential, to use the so-called 1st order formalism where $F^a_{\mu\nu}$ and A^a_μ are treated as independent variables

(7) $$\boxed{S_{\text{free}}[A,F] = \int d^4x \; \frac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu} - \frac{1}{2} F^a_{\mu\nu}\left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu\right)}$$ (← $F_{\mu\nu}$ is not $\partial_{[\mu}A_{\nu]}$ here)

This is totally equivalent to (6) because the path-integral on $F^a_{\mu\nu}$ has no derivatives & it's Gaussian: by completing the square we recover Eq. (6):

gives back original term

$$(8)\quad \frac{1}{4}F^a_{\mu\nu}F^a_{\mu\nu} - \frac{1}{2}F^a_{\mu\nu}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu) = \frac{1}{4}\left(F^a_{\mu\nu} - (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)\right)^2 - \frac{1}{4}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)^2$$

Now, we know that in QED the photon interacts, respecting gauge invariance, to a conserved current J^μ associated a global $U(1)$ symmetry upon which charged matter fields are charged. The J_μ is the Noether current associated to $\psi \to e^{i\alpha}\psi$.

We see now that with several species of massless spin-1 particles we have global symmetries even without introducing external matter fields: just "rotate" among the fields

Global Symmetry

$$(9)\quad \boxed{A^a_\mu \to A^a_\mu - f^{abc}\epsilon^b A^c_\mu \qquad F^a_{\mu\nu} \to F^a_{\mu\nu} - f^{abc}\epsilon^b F^c_{\mu\nu}}$$

with f^{abc} fully antisymmetric

(like eg. for rotations if $a=1,2,3$ $\quad f^{abc} \propto \epsilon^{abc}$ $\quad \vec{A}_\mu \to \vec{A}_\mu - \vec{\epsilon}\wedge\vec{A}_\mu \ldots$)

indeed

antisym.

$$(10)\quad \delta S^{free}[A,F] = \int d^4x\, -\frac{1}{2}\underset{\text{sym in } a\leftrightarrow c}{\cancel{F^a_{\mu\nu}f^{abc}\epsilon^b F^c_{\mu\nu}}} + \underbrace{\frac{1}{2}F^a_{\mu\nu}f^{abc}\epsilon^b\partial_{[\mu}A^c_{\nu]} + \frac{1}{2}f^{abc}\epsilon^b F^c_{\mu\nu}\partial_{[\mu}A^a_{\nu]}}_{\frac{1}{2}F^a_{\mu\nu}\epsilon^b\partial_{[\mu}A^c_{\nu]}(f^{abc}+f^{cba})=0 \text{ antisym.}} = 0$$

This means there is a natural candidate conserved current given by Noether theorem: promote ϵ^a to $\epsilon^a(x)$ dependent on spacetime such that the action is no longer (necessarily) invariant (since now $\partial_\mu \epsilon^a(x) \neq 0$)

Noether Theorem

(11) $$\delta S = \int -\partial_\mu \epsilon^a(x) j_\mu + \epsilon^a(x) \partial_\mu K^a_\mu = \int \epsilon^a(x) \partial_\mu \underbrace{(K^a_\mu + j^a_\mu)}_{J^a_\mu} \underset{\text{e.o.m.}}{=} 0$$

Noether current

by definition of e.o.m.

(12) $$\Rightarrow \boxed{\partial_\mu J^a_\mu \Big|_{\text{e.o.m}} = 0 \quad \text{Noether Theorem}}$$

In the case of sym. (9), by promoting $\epsilon^a \to \epsilon^a(x)$, the $F^a_{\mu\nu}$ that contains no derivatives gives the same transformation, whereas all contribution to δS comes from the $\partial_{[\mu} A^a_{\nu]}$ part of S:

(13) $$\delta S\Big|_{\epsilon(x)} = \int +\frac{1}{2} \underbrace{F^a_{\mu\nu} \partial_{[\mu} \epsilon^b A^c_{\nu]}}_{} f^{abc} \Rightarrow J^b_\mu = -f^{abc} F^a_{\mu\nu} A^c_\nu$$

can remove [,] and multiply by 2

$$\boxed{J^a_\mu = f^{abc} F^b_{\mu\nu} A^c_\nu}$$

conserved current

check:

(1) $\frac{\delta S}{\delta F} = 0 \to F^a_{\mu\nu} = \partial_{[\mu} A^a_{\nu]}$

(2) $\frac{\delta S}{\delta A} = 0 \to \partial_\mu F^a_{\mu\nu} = 0$

e.o.m. from (7)

(14) $$\partial_\mu J^b_\mu = -f^{abc} \Big(\partial_\mu F^a_{\mu\nu} A^c_\nu + \underbrace{\frac{1}{2} F^a_{\mu\nu} \partial_{[\mu} A^c_{\nu]}}_{\text{"0 by e.o.m + antisym. abc}} \Big) \underset{\text{e.o.m.}}{=} 0$$

✓

With the conserved Noether currents in our hand, we can make the obvious guess for the interactions, $A_\mu^a J_\mu^a$. This would be consistent with gauge invariance if, adding $A_\mu^a J_\mu^a$, the J_μ^a remains conserved.

(15) $$S[A,F] = \int d^4x\ \frac{1}{4} F_{\mu\nu}^a F_{\mu\nu}^a - \frac{1}{2} F_{\mu\nu}^a (\partial_\mu A_\nu^a - \partial_\nu A_\mu^a) + \frac{g}{2} A_\mu^a \underbrace{J_\mu^a}_{f^{abc} F_{\mu\nu}^b A_\nu^c}$$

And the J_μ^a remains the conserved current (since we are adding no derivative of fields with this int. term) as long as $A_\mu^a J_\mu^a$ respects the global symmetry. This is so if f^{abc} not only fully antisymmetric but actually satisfy the Jacobi identity.

This is trivial for the SU(2) rotation if $a=1,2,3$ since A_μ^a - rotates & so does J_μ^a. But it's a general fact

(16) $$A_\mu^a J_\mu^a \xrightarrow[\text{Eq.(9)}]{} -\frac{g}{2}\Big[f^{abc} f^{a\bar b\bar c} \epsilon^{\bar b} A_\mu^{\bar c} F_{\mu\nu}^b A_\nu^c + f^{abc} f^{b\bar c\bar d} A_\mu^a \epsilon^{\bar c} F_{\mu\nu}^{\bar d} A_\nu^c + f^{ebc} f^{c\bar b\bar d} \epsilon^{\bar b} A_\mu^a F_{\mu\nu}^b A_\nu^{\bar d} \Big]$$

$$= -\frac{g}{2}\, \epsilon^{\bar b} A_\mu^{\bar c} F_{\mu\nu}^b A_\nu^c \left[f^{abc} f^{a\bar b\bar c} + f^{\bar c a c} f^{a\bar b b} + f^{\bar c b a} f^{a\bar b c} \right]$$

$$= +\frac{g}{2}\, \epsilon^{\bar b} A_\mu^{\bar c} F_{\mu\nu}^b A_\nu^c \Big[\underbrace{f^{bac} f^{\bar b\bar c a} + f^{\bar c a c} f^{b\bar b a} + f^{\bar b c a} f^{\bar c a b}}_{\text{Jacobi id.}\ =\ 0} \Big]$$

So, demanding that f^{abc} are not only antisymmetric but also structure constants we see that $S[A,F]$ has still the global sym. (9) & J_μ^a still conserved.

From Eq. (15)+(16) $\Rightarrow$ $\boxed{\partial_\mu J^a_\mu \big|_{e.o.m.} = 0}$ interacting theory

where

(17) $$\delta S = \int d^4x \; \tfrac{1}{2}\left(F^a_{\mu\nu} - \partial_{[\mu}A^a_{\nu]}\right)\delta F^a_{\mu\nu} + \partial_\mu F^a_{\mu\nu}\,\delta A^a_\nu \quad] \leftarrow \text{from free part}$$

from int.

$$+ \frac{g}{2}\,\delta F^b_{\mu\nu} f^{abc} A^a_\mu A^c_\nu + \frac{g}{2} f^{abc} F^b_{\mu\nu}\left(\underbrace{\delta A^a_\mu A^c_\nu + A^a_\mu \delta A^c_\nu}_{\text{excl. } a\leftrightarrow c \text{ compensated by } \mu\leftrightarrow\nu}\right)$$

so that $\delta S = 0$ gives the e.o.m.

(18a) $$F^a_{\mu\nu} = \partial_{[\mu}A^a_{\nu]} + g f^{abc} A^b_\mu A^c_\nu$$ Yang-Mills e.o.m.

(18b) $$\partial_\mu F^a_{\mu\nu} = + g f^{abc} F^b_{\mu\nu} A^c_\mu = g J^a_\nu$$

Notice that $F^a_{\mu\nu}$ is antisym. $\Rightarrow$ $\partial_\mu \partial_\nu F^a_{\mu\nu} = 0$ (Bianchi identity) is perfectly consistent with $\partial J^a_\nu = 0$ being conserved via Noether Th.
(Exercise: check that indeed $\partial_\mu J^a_\mu \big|_{e.o.m.} = 0$)

The action $S[A,F]$ is still quadratic in F that can be integrated explicitly like before. This corresponds to just plug back the F's e.o.m., indeed from a schematic Lagrangian

(19) $$\mathcal{L}(F,J) = F^T \cdot \frac{K}{2} \cdot F + F^T \cdot J = (F + K^{-1}J)^T \cdot \frac{K}{2} \cdot (F + K^{-1}J) - J^T \frac{K^{-1}}{2} J$$
$(K = K^T)$

so that changing int. variable in the path integral $F \to F - K^{-1}J$ the first term gives just an irrelevant normalization to Z, while the second term just corresponds to insert the e.o.m. solution $KF + J = 0$

$$(20) \qquad \mathcal{L}(F = -K^{-1}J, J) = +J^T \cancel{K^{-1}} \frac{\cancel{K}}{2} K^{-1} J - J^T K^{-1} J = -J^T \frac{K^{-1}}{2} J$$

Doing so to eliminate $F^a_{\mu\nu}$, we get from Eq.(15)+(18a)

$$(21) \qquad S[A, \underbrace{F}_{\partial_{[\mu}A^a_{\nu]} + g f^{abc} A^b_\mu A^c_\nu}] = \int d^4x \, \frac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu}\Big|_{e.o.m.} - \frac{1}{2} F^a_{\mu\nu}\Big|_{e.o.m.} (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu) + \frac{g}{2} A^a_\mu J^a_\mu(F_{e.o.m.}, A)$$

by adding & subtracting $g f^{abc} A^b_\mu A^c_\nu$

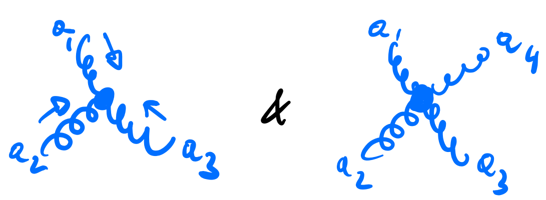
$$= \boxed{\int d^4x \, -\frac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu}\Big|_{F^a_{\mu\nu} = \partial_{[\mu}A^a_{\nu]} + g f^{abc} A^b_\mu A^c_\nu} = S_{YM}[A]}$$

Yang-Mills action

Let's recap what we have done: starting from free collections of spin-1 kinetic terms we observed that a generalization of "flavor rotation" among the A^a_μ provides a conserved current that can be used to build a 3-linear vertex

$\propto f^{abc}$ but since the $F^a_{\mu\nu}$ was just auxiliary field non-propagating, it correspond to $S_{YM}[A]$ with both trilinear & quartic vertices

(22) $\qquad S_{YM} \longrightarrow$



$$(23) \qquad M^{(3)}(1^{a_1} 2^{a_2} 3^{a_3}) \propto g f^{a_1 a_2 a_3} \epsilon_\mu(1) \epsilon_\nu(2) p_3^{[\mu} \epsilon^{\nu]}(3) + \text{permutations}$$

$\longleftarrow$ off-shell

$$\Big| \propto g f^{a_1 b_1 c_1} \left[(\epsilon_1 \cdot p_3)(\epsilon_2 \cdot \epsilon_3) - (\epsilon_2 \cdot p_3)(\epsilon_1 \cdot \epsilon_3) - (\epsilon_3 \cdot p_1)(\epsilon_2 \epsilon_1) \right]$$

$p_1 + p_2 + p_3 = 0$; $p_i^2 = 0$ on-shell

$\epsilon_i \cdot p_i = 0$

Yang-Mills' on-shell 3pt function

check gauge invariance:

(24) $$\epsilon_1 \to \epsilon_1 + p_1 \Rightarrow M^{(3)} \to M^{(3)} + (p_1 \cdot p_3)(\epsilon_2 \cdot \epsilon_3) - (\epsilon_2 \cdot p_3)(p_1 \cdot \epsilon_3) - (\epsilon_3 \cdot p_1)(\epsilon_2 \cdot p_1)$$

$(p_1 \cdot p_3) = 0$: $2 p_1 p_3 = (p_1 + p_3)^2 = p_2^2 = 0$

$+ (\epsilon_3 \cdot p_1)(\epsilon_2 \cdot p_2) = 0$

Non-linear Gauge transformations

The fact that f^{abc} satisfies Jacobi identity calls for a Lie algebra structure, namely introduce some generator T^a with

(25) $$[T^a, T^b] = i f^{abc} T^c \Rightarrow \text{Jacobi id}: 0 = [T^a,[T^b,T^c]] + [T^c,[T^a,T^b]] + [T^b,[T^c,T^a]]$$

(26) $$\begin{cases} A^a_\mu T^a \equiv A_\mu \\ F^a_{\mu\nu} T^a \equiv F_{\mu\nu} \end{cases} \quad \& \quad \begin{cases} A = A_\mu dx^\mu \\ F = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu \end{cases}$$

Lie algebra valued 1-form called "connection" 1-form

← curvature 2-form or field strength

With these names justified by the introduction of a

(27) $$\boxed{D \equiv d - i g A \longleftrightarrow D_\mu \equiv \partial_\mu - i g A^a_\mu T^a}$$

covariant differential

so that acting on D on some 0-form ψ:

(28) $$D\psi = d\psi - igA\psi = (\partial_\mu - igA_\mu^a T^a)\psi \, dx^\mu$$

but acting twice: $D^2\psi = (d - igA)(d\psi - igA\psi) = -ig(dA\,\psi - \cancel{A\,d\psi} + \cancel{A\,d\psi} - igA\wedge A\psi)$

(29) $$\boxed{D^2 = -ig(\underline{dA - igA\wedge A}) = -\frac{ig}{2}(\partial_{[\mu}A_{\nu]} - ig[A_\mu, A_\nu])\,dx^\mu\wedge dx^\nu}$$

(30) $$= -\frac{i}{2}gT^a\underbrace{(\partial_{[\mu}A^a_{\nu]} + gf^{abc}A_\mu^b A_\nu^c)}_{F_{\mu\nu}}\,dx^\mu\wedge dx^\nu = -\frac{ig}{2}F_{\mu\nu}\,dx^\mu\wedge dx^\nu$$

$F_{\mu\nu}$ ← curvature 2-form

This is completely analogous to General Relativity (G.R.)

(31) $$\nabla = d + \Gamma \Rightarrow \nabla^2 = d\Gamma + \Gamma\wedge\Gamma = \frac{1}{2}\underbrace{(\partial_{[\mu}\Gamma^a_{\nu]b} + \Gamma_\mu{}^a{}_c\Gamma_\nu{}^c{}_b - \Gamma_\nu{}^a{}_c\Gamma_\mu{}^c{}_b)}_{R_{\mu\nu}{}^a{}_b}dx^\mu\wedge dx^\nu$$

connection: matrix valued 1-form

$R_{\mu\nu}{}^a{}_b$ Riemann curvature

and like in G.R. $\nabla_\mu\psi$ is covariant, so it's $D_\mu\psi$ w.r.t. gauge group

(31) $$\psi \longrightarrow U(g(x))\psi \qquad U(g(x)) = \exp(i\underbrace{\Omega^a(x)}_{\Omega}T^a)$$

$$D\psi \longrightarrow D'\psi' \equiv U(g(x))D\psi = U(d\psi - igA\psi)$$

$$\| $$

$$(d - igA')U\psi \qquad \text{eq. for } A'$$

$$\Downarrow$$

This is what we mean by gauge covariant

(32) $$\boxed{A \to A' = UAU^{-1} - \frac{i}{g}dU\,U^{-1}}$$

non covariant transf. like connections do

(33) or *infinitesimally*:

$$\boxed{\begin{aligned} A' &= A + i\underbrace{[\Omega, A]}_{\text{adjoint}} + \frac{1}{g}\partial_\mu \Omega = \left(A^a_\mu + \frac{1}{g}\partial_\mu \Omega^a - f^{abc}\Omega^b A^c_\mu\right) dx^\mu \\ &= A^a_\mu + \frac{1}{g}(D_\mu \Omega)^a \end{aligned}}$$

Ω^a in adjoint

which is the non-abelian generalization of the usual gauge transf.

Notice, that the field strength transforms covariantly instead:

(34) $D^2 = -igF$ and $D^2\psi \longrightarrow D'^2\psi' = U D^2 \psi \Rightarrow F' U \psi = U F \psi$

$$\boxed{F' = U F U^{-1}}$$

covariant, in adjoint

(35) $$\boxed{F^a_{\mu\nu} \longrightarrow F^a_{\mu\nu} - \underbrace{f^{abc}\,\Omega^b}_{\text{adjoint rep.}} F^c_{\mu\nu}}$$

infinitesimally
$U = \mathbb{1} + i\Omega^a(x) T^a + \dots$

The Yang-Mills action $S_{YM} = -\int \frac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu}$ can be rewritten as

$(\mathrm{Tr}[T^a T^b] \propto \delta^{ab})$

(36) $$S_{YM} = \int -\frac{1}{2}\mathrm{Tr}[F_{\mu\nu} F^{\mu\nu}] \xrightarrow{\text{gauge}} S_{YM} \quad \text{invariant!}$$

and it's clearly manifestly invariant under gauge transf. (32) i.e. (34)
by cyclicity of trace.

The e.o.m (185) can be put in a manifestly gauge covariant form:

(37) $$\boxed{\partial_\mu F^a_{\mu\nu} - g f^{abc} F^b_{\mu\nu} A^c_\mu = (D_\mu F_{\mu\nu})^a = 0}$$ (with D in adjoint)

$$\partial_\mu F^a_{\mu\nu} - ig[A_\mu, F_{\mu\nu}]^a = \partial_\mu F^a_{\mu\nu} + g A^b_\mu F^c_{\mu\nu} f^{abc} \quad \text{ok}$$

— Quantum Yang-Mills —

The fact that S_{YM} is invariant under gauge transformations

$$(38)\qquad \boxed{A_\mu^a \to A_\mu^a + \frac{1}{g}(D_\mu\Omega)^a; \quad (D_\mu\Omega)^a = \partial_\mu\Omega^a - ig[A,\Omega]^a = \partial_\mu\Omega^a + g f^{abc} A_\mu^b \Omega^c}$$

that is for any function $\Omega^a = \Omega^a(x)$ means that the kinetic term is certainly non-invertible since to any solution e.o.m. A_μ another solution is given by its gauge transformation, even with same initial conditions. This is manifest already in the quadratic part of the action in QED

$$(39)\qquad \underset{\text{photon}}{S^{(2)}} = \int d^4x \, -\frac{1}{4}(\partial_{[\mu}A_{\nu]})^2 = \int d^4x \, \frac{1}{2} A_\mu \underbrace{\left(\eta^{\mu\nu}\Box - \partial^\mu\partial^\nu\right)}_{\Box\underbrace{\left(\eta^{\mu\nu} - \frac{\partial^\mu\partial^\nu}{\Box}\right)}_{\text{projector orthogonal to } \partial^\mu\Omega}} A_\nu$$

$$\left(\eta^{\mu\nu} - \frac{\partial^\mu\partial^\nu}{\Box}\right)\partial_\nu\Omega = 0$$

In the abelian case, this is all there is, and one can have a covariant propagator by fixing a gauge via the path-integral.

— Faddeev-Popov Trick —

we can add to the path integral an explicit integration over the gauge-dependent part of A_μ since S_{YM} is indep., this may only change the irrelevant normalization of Z.

for arbitrary $w(x)$

(40) $$Z \propto \int [dA] \int [d\Omega]\, \delta(\Omega(x) - w(x)) \cdot \exp \underbrace{[i S[A]]}_{\text{indep. of } \Omega}$$

and we can equally write $\Omega(x) = w(x)$ as the solution of $G(A_\Omega) = 0$ eq.

gauge fixing condition

(41) $$\boxed{Z \propto \int [dA] \int [d\Omega]\, \delta(G(\underset{A_\mu - \frac{1}{g}\partial_\mu \Omega}{A_\Omega}))\, \det\left(\frac{\delta G(A_\Omega)}{\delta \Omega}\right) \exp[i S[A]]}$$

for some $G(A_\Omega)$ where A_Ω is the gauge-transf field, $A_\Omega = A_\mu - \frac{1}{g}\partial_\mu \Omega$

To be specific, let's pick one particular family of gauge:

choose gauge

(42) $$\boxed{G(A) = \partial_\mu A^\mu - w(x)} \Rightarrow \boxed{\frac{\delta G(A_\Omega)}{\delta \Omega} = -\frac{1}{g} \Box \delta^4(x-y) \Rightarrow \det(\ ) = \text{const}}$$

just changes normalization in abelian case

(43) $$Z \propto \int [dA][d\Omega]\, \delta(\partial_\mu A^\mu - \frac{1}{g}\Box\Omega - w(x)) \exp(i S[A])$$

$$= \int [dA][d\Omega]\, \delta(\partial_\mu A^\mu - w(x)) \exp(i S[A])$$

$A_\mu \to A_\mu + \partial_\mu \frac{\Omega}{g}$ change of variables (and by assumption of gauge invariance $[dA] \exp[i S[A]]$ is invariant)

Moreover, everything here is true for any $w(x)$ so that we could equally integrate over some weight $F[w]$

(44) $$Z \propto \int [dA][d\Omega][dw]\, F[w]\, \delta(\partial_\mu A^\mu - w(x)) \exp(i S[A])$$

$$= \int [dA][d\Omega]\, F[\partial_\mu A^\mu] \exp(i S[A]) \propto \int [dA]\, F[\partial_\mu A^\mu] \exp(i S[A])$$

no longer matter: we dump it into the irrelevant normalization choosing finally a Gaussian $F[\omega]$,

$$F[\omega] = \exp\left(-\frac{i}{\xi}\int \frac{\omega^2(x)}{2}\, d^4x\right) \tag{45}$$

$$\Downarrow$$

Fixing Gauge in QED with Path integral

$$Z \propto \int [dA] \exp\left[i\int d^4x \left(-\frac{(F_{\mu\nu})^2}{4} - \frac{1}{2}\frac{(\partial_\mu A^\mu)^2}{\xi}\right)\right] \tag{46}$$

This now corresponds to the abelian gauge theory with propagator

$$\langle \hat{A}_\mu(k)\, \hat{A}_\nu(p)\rangle = (2\pi)^4 \delta^4(k+p)\cdot \frac{-i\left(\eta^{\mu\nu} - (1-\xi)\, p^\mu p^\nu/p^2\right)}{p^2+i\epsilon} \tag{47}$$

($\xi = 1$ Feynman gauge, $\xi = 0$ Landau gauge)

— Faddeev – Popov non-Abelian —

Now, the non-abelian case is completely analogous but the determinant in Eq. (42) is A_μ-dependent, and this brings the appearance of "Ghosts"

anticommuting scalars

From (41): repeating previous steps

$$Z \propto \int [d\omega] \int [dA] \int [d\Omega]\ \overset{\text{weight}}{F[\omega]}\ \overbrace{\delta\left(G^a(A_\Omega)\right) \det\left(\frac{\delta G^a(A_\Omega)}{\delta \Omega^b}\right)}^{\text{Gauge-fixing}} \exp\left[i\, S_{YM}[A]\right] \tag{48}$$

$$A_\Omega = UAU^{-1} - \frac{i}{g}\, dU\, U^{-1} = \left(A^a_\mu + \frac{1}{g}(D_\mu\Omega)^a\right) T^a$$

$$(D_\mu \Omega)^a = \partial_\mu \Omega^a + g f^{abc} A^b_\mu \Omega^c$$

depends on A

$$U = \exp[i\Omega^a(x)\, T^a]$$

$$(49)\begin{cases} G^a(A_\Omega) = \partial_\mu A^{\mu a}_\Omega - \omega^a(x) = \partial_\mu A^a_\mu + \frac{1}{g}\partial_\mu (D_\mu \Omega)^a - \omega^a(x) \\ \boxed{\frac{\delta G^a(A_\Omega)}{\delta \Omega^b} = \frac{\partial^\mu}{g}\left(\partial_\mu \delta^{ab} - g f^{abc} A^c_\mu\right)\delta^4(x-y) \equiv \frac{1}{g}\partial_\mu D_\mu(A)} \end{cases}$$

where now $\det\left(\frac{\delta G^a}{\delta \Omega^b}\right) = \det\left(\frac{1}{g}\partial_\mu D_\mu(A)\right)$ is a A-functional

$$(50)\quad \mathcal{Z} \propto \int [dA]\, F[(\partial_\mu A^{a\mu})]\, \det\left(\frac{1}{g}\partial_\mu D_\mu(A)\right) \exp\left(i S_{YM}[A]\right)$$

changing variables
$A_\Omega \to A$

Taking the F Gaussian as before we have then

difference w.r.t. QED

$$(51)\quad \boxed{\mathcal{Z} \propto \int [dA] \exp\left[i S_{YM}[A] - \int \frac{i}{2\xi}(\partial_\mu A^a_\mu)^2\right] \det\left(\frac{1}{g}\partial_\mu D_\mu(A)\right)}$$

Now, the determinant can also be written as a local term in the Lagrangian in terms of anticommuting c & $\bar{c}$ variables known as ghost fields

$$(52)\quad \boxed{\begin{aligned} \mathcal{Z} &\propto \int [dc\, d\bar{c}][dA] \exp\left[i S_{YM}[A] - \frac{i}{2\xi}\int d^4x\, (\partial_\mu A^a_\mu)^2 - i\int \bar{c}^a \partial^\mu D^{ab}_\mu(A)\, c^b\, d^4x\right] \\ &= \int [dc\, d\bar{c}]\,[dA] \exp\left[i\int d^4x\, -\frac{1}{4}(F^a_{\mu\nu})^2 - \frac{1}{2\xi}(\partial_\mu A^a_\mu)^2 + \partial_\mu \bar{c}^a\left(\delta^{ab}\partial^\mu - g f^{abc} A^c_\mu\right) c^b\right] \end{aligned}}$$

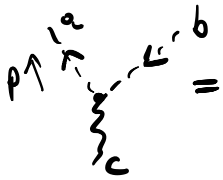
so that the propagators are

$$(53)\quad \begin{cases} \langle \hat{A}^a_\mu(k)\, \hat{A}^b_\nu(p) \rangle = \dfrac{-i\,(\eta^{\mu\nu} - (1-\xi)\, p^\mu p^\nu)}{p^2 + i\epsilon}\, \delta^{ab} \\[2ex] \langle \hat{c}^a\, \hat{\bar{c}}^b \rangle = \dfrac{i}{p^2 + i\epsilon}\, \delta^{ab} \end{cases}$$

Ghost propagator

and in addition to the vertices of YM there are couplings to the Ghosts

(54)



$$= -g\, p_\mu\, f^{abc}$$

Ghost vertex with A_μ